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Tensor product decomposition and CG coefficients for nonsemisimple Hopf algebras

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Abstract. The tensor product of two $U_q sl(2)$ modules, q a root of 1, is decomposed into indecomposable summands for both irreducible and *indecomposable* modules. Clebsch–Gordan coefficients in the general case are computed. An apparently new identity is derived and some possible applications are conjectured.

1. Tensor product decomposition

This paper follows on from [1] (referred to as I) on the structural theory of a class of nonsemisimple Hopf algebras. Here we focus on tensor products of modules. In particular, we derive a complete result on tensor product decomposition of two $U_q sl(2)$-modules, q a pth root of 1 for some odd p. The Clebsch–Gordan (CG) coefficients in the general case of $U_q sl(2)$ indecomposable modules are also computed. As to be expected they have a rather complicated structure. First we fix some notation and choice of basis. We use the standard symbols and formulae (see I) for the generators of $U_q sl(2)$ and its bialgebra structure. Let V_m denote the irreducible module generated by a singular vector x_m of weight q^m, $m \leqslant p-1$. Let Q_n denote the projective indecomposable module (p.i.m.) generated by a vector y_n of weight q^{-n}. The following basis for Q_n will be chosen. Let

$$\alpha_n \equiv \frac{E^{n-1}}{[n-2]!} \cdot y_n. \tag{1}$$

Then

$$F^{(k)}\alpha_n \equiv \frac{F^k}{[k]!}\alpha_n \qquad \text{and} \qquad \frac{F^{(r)}}{[r]!} \cdot y_n$$

form a basis for Q_n. Furthermore, α_n and $F^{(n-1)} \cdot \alpha_n$ are E-singular vectors in Q_n. We call these parent and daughter, respectively. We have

$$E y_n = F^{(n-2)}\alpha_n \tag{2}$$

$$EF^{(k)} y_n = [1-k-n]F^{(k-1)} \cdot y_n + \begin{bmatrix} n+k-2 \\ k \end{bmatrix} F^{(n+k-2)}\alpha_n. \tag{3}$$

The following result is known. With the notation as above

$$V_m \otimes V_n = \bigoplus_{k=|m-n|,+2,+4,\ldots}^{\min\{(m+n),2p-m-n-2\}} V_k \bigoplus_{r=s,s+2,\ldots}^{m+n+2-p} X_r \tag{4}$$

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where $r = 1$ if $m + n - p$ odd, otherwise $r = 2$ and X_r is Q_r if $r \geqslant 2$ and $X_1 = P$.

As for the tensor product of type $Q_m \otimes V_n$ or $Q_m \otimes Q_n$ it is only known that they are the sum of P and Q_n's. This follows easily from the analysis of I since it is known that Q_n and P are the only p.i.ms and since projective $\otimes$ anything is projective. Our aim in this section is to determine exactly which p.i.m. occur in the product. It is seen from the stucture of Q_n and a simple consideration of dimensions that the total number of independent E-singular vectors in $Q_m \otimes V_n$ equals $2(m+1)$. If there are k summands of P type and r summands of Q type then $2k + r = 2(m+1)$. Our strategy is to find a set of singular vectors first and then pair them in different Q_n. As we see below this is not so straightforward. We suppress the subscript n for convenience.

Let α and y be generators Q_n with α a singular vector as in the last paragraph. Let x be the singular vector in V_m. Consider a K-eigenvector φ_N in the product with eigenvalue $q^{m+n-2(N+1)}$. Let

$$\varphi_N = \sum_{k=0}^{N} c_k F^{(k)} \alpha \otimes F^{(N-k)} x \oplus \sum_{s=0}^{N-n+1} d_s F^{(s)} y \otimes F^{(N-n-s+1)} x. \tag{5}$$

φ_N is a singular vector if and only if the following recursion relations hold

$$[n-k-1] c_{k+1} = -q^{n-2(k+1)} [m-N+k+1] c_k \qquad k < n-2 \tag{6}$$

$$[s+n] d_{s+1} = q^{-(n+2s)} [m-N+s+n] d_s \tag{7}$$

$$[s] c_{n+s-1} - \begin{bmatrix} n+s-2 \\ s \end{bmatrix} d_s = q^{-(n+2s-2)} [m-N+n+s-1] c_{n-2+s}. \tag{8}$$

Before solving these equations let us make some observations which follow as very simple consequences. First, $d_0 \propto c_{n-2}$ and if d_0 (hence all $d_i = 0$) then $c_k = 0$ for $k \leqslant n-2$. Thus we get two sequences of singular vectors $\varphi_N^{(1)}$ which start with α and $\varphi_N^{(2)}$ starting with $F^{(n-1)}\alpha$ in the first factor. In the case of $\varphi_N^{(2)}$ the above relations are slightly modified if $p - n < N \leqslant m$. In that case $d_{p-n+1} \propto b_{p-n}$ where $b_k = c_{n-2+s}$. In any case it is seen that there are precisely $2(m+1)$ singular vectors. One writes the singular vectors and then determines which ones pair up. A singular vector, φ_N, with weight q^{k-2}, $k = m + n - 2 - 2N$ pairs with another singular vector, φ_M, with weight q^{-k} and $\varphi_M \propto F^{k+1} \cdot \varphi_N$. However, if k exceeds p then we must take $k \bmod p$ and for the case where k is negative we may need to consider $p - k$. Thus, the weight table alone is not sufficient. It may be checked whether φ_M descends from φ_N and this can be done by comparing the ratio of coefficients given above. Furthermore in (5) the coefficients c_0 and c_{n-1} are arbitrary. Note that c_{n-1} appears only if $m > n-2$. Moreover, if $N \geqslant n-1$ then $\varphi_N^{(1)}$ is not unique since a singular vector from the second class can always be added to the former. We now have the following result.

Lemma 1. *The coefficients c_k and d_s of the first series of singular vectors φ_N (starting with α) are given by*

$$c_k = (-1)^k \frac{[m-N+k]![n-2-k]!}{[n-2]![m-N]!} q^{k(n-k-1)} \qquad k \leqslant n-2 \tag{9}$$

$$d_s = (-1)^{n-1} \frac{[m-N+s+n-1]![n-1]}{[s+n-1]![m-N]!} q^{-s(n+s-1)} c_0 \tag{10}$$

and

$$c_{n-1+s} = \frac{[m-N+n+s-1]!}{[s]!} q^{-s(n+s-1)} \sum_{r=0}^{s} \gamma_{sr} \tag{11}$$

where

$$\gamma_{s0} = \frac{c_{n-1}}{[m-N+n-1]!}$$

and

$$\gamma_{sr} = \frac{[n-1]}{[n-2]![m-N]![r][n-1+r]} c_0.$$

We are now in a position to state the general theorem on tensor product decomposition. One uses the structural results of I to see which singular vectors pair up.

Theorem 1. *Let Q_n and V_m be as above.*

(1) Let $m+n \leqslant p$ and consider two cases separately.

(i) $m \leqslant n-2$, then

$$Q_n \otimes V_m = Q_{m+n-2} \oplus Q_{m+n-4} \oplus Q_{m+n-6} \oplus \cdots \oplus Q_{n-2-m}. \quad (12)$$

(ii) $m = n-2+r$, $r \geqslant 1$,

$$Q_n \otimes V_m = Q_{m+n-2} \oplus Q_{m+n-4} \oplus \cdots Q_r \oplus 2Q_{r-2} \oplus 2Q_{r-4} \oplus \cdots 2Q_2(2P) \quad (13)$$

where the module P occurs if $m+n$ is odd. The number 2 denotes multiplicity.

(2) $m+n = p+r$, $r \geqslant 1$. Here we also consider two cases.

(i) $m \leqslant n-2$

$$Q_n \otimes V_m = 2Q_r \oplus 2Q_{r-2} \oplus \cdots \oplus 2Q_2(2P) \oplus Q_{p-r} \oplus Q_{p-r-2} \oplus \cdots \oplus Q_{n-m-2}. \quad (14)$$

(ii) $m = n-2+s$

$$Q_n \otimes V_m = 2Q_r \oplus 2Q_{r-2} \oplus \cdots \oplus 2Q_2(2P) \oplus 2Q_{m-n} \oplus 2Q_{m-n-2} \cdots \oplus 2Q_2(2P) \oplus Q_{p-r-2} \oplus Q_{p-r-4} \oplus \cdot \oplus Q_{m-n+2}. \quad (15)$$

It was assumed that $m \leqslant p-2$. For the special case there is slight modification in the formulae.

We illustrate the intricate behaviour by a visual presentation of the relationships. Recall that we have two series $\varphi^{(i)}$, $i = 1, 2$ of singular vectors. These two are written in columns according to the eigenvalues of $\ln K$. The descendants are on the right or below and shown by an arrow between the two columns and the kinships within a column are indicated by brackets. The $\varphi^{(2)}$ series is on the right. We demonstrate only for the cases (1)(i) and (2)(ii). For (1)(i) we have the following:

$$\begin{array}{lllll}
1 & m+n-2 & \longrightarrow & -(m+n) & F^m \\
F & m+n-4 & \longrightarrow & -(m+n-2) & F^{m-1} \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
F^m & n-m-2 & \longrightarrow & m-n & 1
\end{array}$$

For (2)(ii) the kinship diagram is given below. Note that the columns within square brackets are paired by nested brackets e.g. F and F^{r-2}.

$$\left[\begin{array}{llcll}
1 & p+r-2 & \cdot & m+n-2p-2 & F^{p-n+1} \\
F & p+r-4 & \cdot & m+n-2p-4 & F^{p-n+2} \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
F^{r-2} & p-r+2 & \cdot & -(m+n-2) & F^{p-n+r-1} \\
F^{r-1} & p-r & \cdot & -(m+n) & F^{p-n+r}
\end{array}\right]$$

$$\begin{array}{lllll} F^r & p-r-2 & \longrightarrow & -(p-r) & F^{p-n} \\ F^{r+1} & p-r-4 & \longrightarrow & -(p-r-2) & F^{p-n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F^{n-2} & m-n+2 & \longrightarrow & -(m-n+4) & F^{m-n+2} \end{array}$$

$$\begin{bmatrix} F^{n-1} & m-n & & m-n & 1 \\ F^n & m-n-2 & & m-n-2 & F \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F^{m-1} & n-m & & n-m & F^{m-n} \\ F^m & n-m-2 & & n-m-2 & F^{m-n+1} \end{bmatrix}$$

The result on tensor product of two Q-type is as follows.

Theorem 2.

$$Q_n \otimes Q_m = 4P \oplus \sum_{r=2}^{p} 4Q_r.$$

The kinship diagrams in this case are complicated. In any case the diagrams above are the starting point for deriving CG coefficients in the general case.

2. CG coefficients

In this section we calculate the CG coefficients for the product $Q_n \otimes V_m$. These coefficients are known for the product $V_k \otimes V_m$ [2]. From lemma 1 and theorem 1 we see that the cases (1)(ii) and (2)(ii) are relatively complicated as they involve the generator y_n. We consider only the case (1)(ii), the others can be handled similarly. Note that the modules Q_k are not generated by the singular vectors alone. We need to know the corresponding y_k. Thus let $m+n-1 \leqslant p$ and $m \geqslant n-1$. For each $N \leqslant m$, let $M = m+n-2N$ and α_M be the corresponding parent singular vector. The module Q_M is generated by α_M and the respective y_M. Our first task is to compute y_M. Since we are dealing with singular vectors from the first series we write $y_M^{(1)}$. For $M \geqslant 2$ let

$$\alpha_M = \sum_{k=0}^{N} c_k F^{(k)}\alpha \otimes F^{(N-k)}x + \sum_{s=0}^{N-n+1} d_s F^{(s)} y \otimes F^{(N-n+1-s)}x. \tag{16}$$

Then with

$a = \max(0, m+n-2-N)$

$$F^{(M-2)} \cdot \alpha_M = \sum_{s=a}^{M+N-2} g_s F^{(s)}\alpha \otimes F^{(M+N-2-s)}x + \sum_{s=0}^{M+N-n-1} h_s F^{(s)} y \otimes F^{(M+N-n-1-s)}x \tag{17}$$

where

$$g_s = \sum_{r=0}^{s} c_r \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M+N-2-s \\ N-r \end{bmatrix} q^{(s-r)(n-2-r-s)} \tag{18}$$

and

$$h_s = \sum_{0}^{s} d_r \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} m-N-s \\ N-n-r+1 \end{bmatrix} q^{-(s-r)(n+r+s)}. \tag{19}$$

Note that if we replace M by $M+1$ in (17) then we get the corresponding daughter singular vector. Comparing with the expressions (10) and (11) we obtain some summation formulae one of which appears to be rather strange. I discuss this in the appendix. Returning to the generator y_M let

$$y_M = \sum_{a+1}^{M+N-1} p_r F^{(r)}\alpha \otimes F^{(M+N-1-r)}x + \sum_{0}^{M+N-n} t_r F^{(r)} y \otimes F^{(M+N-n-r)}x. \tag{20}$$

Now

$$E \cdot y_M^{(1)} = F^{(M-2)} \cdot \alpha_M \tag{21}$$

yields the following set of equations:

$$p_r[n-r-1] + p_{r-1}[N-n+r+1]q^{n-2r} = g_{r-1} \qquad \text{for} \quad a < r \leqslant n-2 \tag{22}$$

$$-[n-1+r]t_r + [N+r]t_{r-1}q^{-(n+2(k-1))} = h_{r-1} \tag{23}$$

$$p_{n-1+r}[r] - t_r \begin{bmatrix} n+r-2 \\ r \end{bmatrix} - [N+r]p_{n-2+r} = -g_{n-2+r}. \tag{24}$$

The solution of these inhomogeneous difference equations is straightforward. Thus

$$p_r = \sum_{a+1}^{r} (-1)^{r-a+1} \frac{[N-n-1+r]![n-2-r]!}{[N-n-1+s]![n-1-s]!} q^{(r-s)(n-r-s-1)} g_{s-1} \qquad r \leqslant n-2 \tag{25}$$

$$t_r = -\sum_{0}^{r} \frac{[N+r]![n-2+k]!}{[N+k]![r+n-1]!} q^{-(r-k)(n-1+r+k)} h_{k-1} \tag{26}$$

with $h_{-1} = [n-1]t_0$ and

$$\begin{aligned} t_0 &= g_{n-2} - q^{-(n-2)}[N]p_{n-2} \\ \sigma_0 &= p_{n-1} \\ \sigma_k &= \frac{\begin{bmatrix} n+k-2 \\ k \end{bmatrix} t_k - g_{n-2+k}}{[k]} \\ p_{n-1+r} &= \sum_{k=0}^{r} \frac{[N+r]![k]!}{[N+k]![r]!} q^{-(r-k)(n-2+r+k+1)} \sigma_k. \end{aligned} \tag{27}$$

We are now in a position to calculate the CG coefficients. We fix some notation. Recall that we are dealing with case (1)(ii) of theorem 1 when $m > n-2$ and $m+n \leqslant p$. Corresponding to a module Q_k in the tensor product decomposition we still use the symbols α_k and y_k for the two generators. There is no confusion since for the original modules Q_n and V_m we write the respective generators without subscripts. Set $M = m+n-2N$ and $N+L = T$, let

$$\begin{aligned} F^{(L)} \cdot \alpha_M = {} & \sum_s C(M, L, s, T-s) F^{(s)}\alpha \otimes F^{(N+L-s)}x \\ & + \sum D(M, L, s, T+1-n-s) F^{(s)} y \otimes F^{(N+L+1-n-s)}x \end{aligned} \tag{28}$$

and

$$\begin{aligned} F^{(L)} \cdot y_M = {} & \sum A(M, L, s, M+L-s-1) F^{(s)}\alpha \otimes F^{(M+L-s-1)}x \\ & + \sum B(M, L, s, M+L-n-s) F^{(s)} y \otimes F^{(M+L-n-s)}x. \end{aligned} \tag{29}$$

From theorem 1 we see that the modules Q_k, $k = m-n+2, m-n, \ldots, 2$ occur twice corresponding to the two classes of singular vectors. We distinguish the singular vectors, CG coefficients, etc, of the second class (those starting with $F^{n-1+k}\alpha$ in the first factor) by inserting a prime.

The coeffcients $C(M, L, \_, \_)$ etc, in the two equations above are the generalized CG coefficients. First C and D are computed from the expressions (9)–(11) for the coefficients of singular vectors. We can now state the following.

Theorem 3. *The CG coefficients in the decomposition of $Q_n \otimes V_m$ into indecomposable summands are as follows. With notation as above and $M = m + n - 2N$ and $T = N + L$*

$$C(M, L, s, T - s) = q^{sX} \sum_0^s (-1)^r \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} N + L - s \\ N - r \end{bmatrix} \frac{[m - N + r]![n - 2 - r]!}{[n - 2]![m - N]!} q^{-rY} c_0 \tag{30}$$

with

$$X = L + 2N - s - m \qquad \textit{and} \qquad Y = L + 2N + 1 - n - m$$

and

$$D(M, L, s, T + 1 - n - s) = (-1)^{n-1}[n - 1]q^{s(X+2-2n)} \sum_0^s q^{-r(Y+2)} \begin{bmatrix} s \\ r \end{bmatrix} \times \begin{bmatrix} T + 1 - n - s \\ N + 1 - n - r \end{bmatrix} \frac{[M + N + r - 1]!}{[r + n - 1]!} c_0. \tag{31}$$

To compute A and B we use expressions (25)–(27). Thus

$$A(M, L, s, M + L + N - 1 - s) = \sum_{a+1}^s q^{(s-r)U} \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M + L + N - 1 - s \\ M + N - 1 - r \end{bmatrix} p_r \tag{32}$$

with

$$U = L + 2(M + N - 1) - m - r - s$$

and

$$B(M, L, s, M + L - n - s) = \sum_0^s q^{(s-r)(U-2(N+n-2))} \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M + L - n - s \\ M - n - r \end{bmatrix} t_r. \tag{33}$$

The coefficients for the second class of p.i.m. for $M = m - n + 2 - 2N$

$$C'(M, L, s, L + N + n - 1 - s) = q^{sX} \sum_k q^{-kY} \begin{bmatrix} n - 1 + s \\ n - 1 + k \end{bmatrix} \times \begin{bmatrix} L + N - s \\ s - k \end{bmatrix} \begin{bmatrix} m - N + k \\ k \end{bmatrix} \tag{34}$$

$$A'(M, L, s, M + L + N - 1 - S) = \sum q^{(s-r)U} \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M + L + N - 1 - s \\ M + N - 1 - r \end{bmatrix} p'_r \tag{35}$$

with

$$p'_r = (-)^r \sum_{k=n}^r (-1)^s \frac{[m - N + r + 1]![n - 2 - r]!}{[m - N + k + 1]![n - 1 - k]!} \beta_{k-1} \tag{36}$$

and

$$\beta_k = q^{k(M-k+2N-m)} \sum q^{-r(M+2N-m+n-1)} \begin{bmatrix} n - 1 + k \\ n - 1 + r \end{bmatrix} \begin{bmatrix} M + N - k \\ k - r \end{bmatrix} \begin{bmatrix} m - N + k \\ k \end{bmatrix}. \tag{37}$$

The CG coefficients can be put into a more tractable form and can often be identified with some known functions like the basic hypergeometric functions. However, the formulae for A and B look rather formidable. Since Q_n can be obtained from the product of two *irreducible* modules the coefficients derived above can be related to 6-j symbols coresponding to the product of three irreducible modules. The structure of the tensor product of modules of higher quantum algebras is much more involved. Only some partial results are known (see [4]) for $U_q sl(3)$.

3. An identity

We discuss some identities in this section. These can be derived from the fact that for a parent singular vector α_M in the product $F^{M-2} \cdot \alpha_M$ is also a singular vector and hence must be proportional to a daughter singular vector. Identifying the latter from the diagrams in section 1 we obtain the required identity. In the case $d_s = 0$ in (5) we get the q-Saalschütz identity [3]. However, if $d_s \neq 0$ then some apparently new identities appear. We mention one. Let $M = m + n - 1 - N$:

$$\sum_{r=0}^{n-2} (-1)^r \begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M-s \\ N-r \end{bmatrix} \frac{[n-2-r]!}{[n-2]!} \frac{[m-N+r]!}{[m-N]!} c_0 + \sum_{r=n-1}^{s} \left(\begin{bmatrix} s \\ r \end{bmatrix} \begin{bmatrix} M-s \\ N-r \end{bmatrix} \frac{[M+r]!}{[s]!} \right.$$
$$\left. \times \left(\frac{c_{n-1}}{[M]!} + \frac{[n-1]c_0}{[n-2]![m-N]!} \sum_{k=1}^{r} \frac{1}{[k][n+k-1]} \right) \right)$$
$$= \frac{[N+s]!}{[s]!} \left(\frac{a}{[N]!} + \frac{b}{[N-n+1]!} \cdot \sum_{k=1}^{s} \frac{1}{[n-2]![k][n-1+k]} \right).$$

Here a and b are constants which can be determined easily. This identity can be proved directly. First, one puts it in the form of a terminating basic hypergeometric series. The first sum S_1 on the left-hand side can be identified with a ${}_4\phi_3$ series [3]. As for the second sum S_2 the first part (without the inner sum) can be summed. S_1 is then transformed using Sear's formula [3]. The inner sum in S_2 is also modified so that S_2 becomes the sum over two independent indices. Then, by adding appropriate terms of S_1 and S_2, we get the right-hand side of the above identity. We omit the details. However, one wonders whether the proof can be simplified. Unfortunately, due to limited knowledge of the basic hypergeometric series the author was unable to do this. This identity seems to be new and more identities may be deduced by considering the product of two indecomposable modules.

4. Discussion

As mentioned in I the indecomposable modules may describe some metastable or unstable states. The generalized CG coefficients may then be used for the description of composite systems. We note the following fact which seems physically obvious. The composite state of two systems, one stable and the other metastable, is metastable. Moreover, note that a small perturbation in the parameter q will decompose the indecomposable module into irreducible ones corresponding to stable states. The preceding discussion is admittedly vague and conjectural but I feel that it may be put in more rigorous form by considering specific models.

References

[1] Patra M K 1999 On the structure of nonsemisimple Hopf algebras *J. Phys. A: Math. Gen.* **32** 159

[2] Kirillov A N and Reshetikhin N Yu 1989 *Infinite Dimensional Lie Algebras and Groups* ed V G Kac (Singapore: World Scientific)

[3] Gasper G and Rahman M 1990 *Basic Hypergeometric Series (Encyclopedia of Mathematics and its Applications)* (Cambridge: Cambridge University Press)

[4] Patra M K 1997 *J. Phys. A: Math. Gen.* **30** 1259